UDC 531.36

ON PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDERS

PMM Vol. 41, № 2, 1977, pp. 362-365
L. A. KIPNIS
(Voronezh)
(Received February 9, 1976)

The existence and uniqueness of a T-periodic solution of the nonlinear differential equation

$$d^n x / dt^n + f(t, x) = 0 \quad (n \geqslant 3) \tag{1}$$

is proved, and stability of the solutions of the equivalent system

$$\begin{gathered} dz / dt = F(t, z) \\ z = (z_1, z_2, \ldots, z_n), \quad F = (F_1, F_2, \ldots, F_n), \quad z_1 = x, \ F_i = z_{i+1} \\ (i = 1, 2, \ldots, n-1), \quad F_n = -f(t, z_1) \end{gathered} \tag{2}$$

is studied. In what follows, E_n denotes an n-dimensional Euclidean space of elements z with the scalar product

$$(z, h) = \sum_{i=1}^{n} z_i h_i \quad (z, h \in E_n, \ \| z \| = (z, z)^{1/2})$$

The following theorem holds.

Theorem 1. Let the function $f(t, x)$ satisfy the following conditions:

1) f and $\partial f / \partial x$ are continuous for all $t, x \in (-\infty, \infty)$;

2) a number T exists such that $f(t + T, x) \equiv f(t, x)$ for all t and x;

3) the inequality $a \leqslant \partial f / \partial x \leqslant b$, where a and b are constants, holds for all t and x Then in each of the following cases:

a) $n = 2k + 3 \quad (k = 0, 1, 2, \ldots), \quad ab > 0$;

b) $n = 4k + 4 \quad (k = 0, 1, 2, \ldots), \quad a > 0, \ b > 0$;

c) $n = 4k + 6 \quad (k = 0, 1, 2, \ldots), \quad a < 0, \ b < 0$

the equation (1) has a unique T-periodic solution.

Proof. From [1] it follows that the sufficient condition for a unique T-periodic solution of the system (2) to exist is, that the conditions

$$(-(U[\partial F / \partial z] + [\partial F / \partial z]' U)h, h) \geqslant \| h \|^2 \tag{3}$$

$$\| F(t, z) - F(t, h) \| \leqslant L \| z - h \|, \quad 0 < L = \text{const} \tag{4}$$

hold for all $t, x \in (-\infty, \infty)$, $z, h \in E_n$. Here U is a symmetric reversible matrix with both positive and negative eigenvalues, and the matrix $[\partial F / \partial z]'$ is a transposition of $\partial F / \partial z$. The system (2) has no other restrictions, provided that the conditions (3) and (4) both hold.

The condition (4) obviously follows from the inequality (3) of the theorem. We shall show that (3) automatically implies that the matrix U is reversible, as well as the fact that it has both positive and negative eigenvalues. Indeed, writing for any t_0, z° $\partial F(t_0, z^\circ) / \partial z = A$, we have

$$\| h \|^2 \leqslant (-(UA + A'U)h, h) \leqslant 2 \| A \| \ \| h \| \ \| Uh \|$$

therefore $\| Uh \| \geqslant \| h \| / (2 \| A \|)$. From here it follows that U is reversible. It is evident that the matrix $A = (a_{ij})$ $(i, j = 1, 2, \ldots, n)$ has the form $a_{n1} = -p_0 = -\partial f(t_0, z_1^\circ) / \partial z_1$, $a_{ii+1} = 1$ $(i = 1, 2, \ldots, n-1)$, with the remaining a_{ij} equal to zero, $z^\circ = (z_1^\circ, z_2^\circ, \ldots, z_n^\circ)$.

According to the condition of the theorem we can either have $p_0 > 0$ or $p_0 < 0$ only. Let $p_0 > 0$. The eigenvalues of the matrix A can be obtained from the equation $l^n + p_0 = 0$ and are equal to

$$l_k = \sqrt[n]{p_0}\left[\cos \frac{\pi + 2\pi k}{n} + i \sin \frac{\pi + 2\pi k}{n}\right] \qquad (k = 0, 1, 2, \ldots, n-1)$$

Let us assume that the spectrum of U is positive. Then condition (3) implies that $\mathrm{Re}\, l_k < 0$ $(k = 0, 1, \ldots, n-1)$ (see [2]) and this is impossible since $\mathrm{Re}\, l_0 > 0$. Assume now that the spectrum of U is negative. Condition (3) implies that $\mathrm{Re}\, l_k > 0$. Indeed, if h_k is the eigenvector of the matrix A corresponding to l_k, then we have

$$\begin{aligned}(-(UA + A'U)h_k, h_k) &= (-UAh_k, h_k) + (-Uh_k, Ah_k) = \\ l_k(-Uh_k, h_k) + \bar{l}_k(-Uh_k, h_k) &= (2\mathrm{Re}\, l_k)\cdot(-Uh_k, h_k) \geqslant \| h_k \|^2\end{aligned}$$

where $\bar{l}_k$ is a conjugate of l_k. Since $(-Uh_k, h_k) > 0$, we have

$$2\mathrm{Re}\, l_k \geqslant \| h_k \|^2 / (-Uh_k, h_k) \geqslant \| h_k \|^2 / (\| U \| \; \| h_k \|^2) = 1 / \| U \|$$

On the other hand, if an integer k is chosen so that $\frac{1}{4}n - \frac{1}{2} < k < \frac{3}{4}n - \frac{1}{2}$, then $\cos(\pi + 2\pi k)/n < 0$, which contradicts the condition that $\mathrm{Re}\, l_k > 0$. The case $p_0 < 0$ is considered in the same manner.

It follows therefore that the matrix U has both positive and negative eigenvalues. To complete the proof of the theorem it remains to show that the matrix U satisfies the condition (3) which implies the nonnegative definiteness of the matrix $B = (b_{ij})$ $(i, j) = (1, \ldots, n)$ of the form

$$\begin{aligned}&b_{11} = 2pu_{1n} - 1, \quad b_{ii} = -2u_{i-1i} - 1 \\ &(i = 2, 3, \ldots, n), \quad b_{1j} = pu_{jn} - u_{1j-1} \quad (j = 2, 3, \ldots, n) \\ &b_{ij} = -(u_{ij-1} + u_{i-1j}) \quad (i = 2, 3, \ldots, n-1;\; j = i+1, i+2, \ldots, n)\end{aligned}$$

where $p = \partial f(t, z_1) / \partial z_1$, u_{ij} $(i, j = 1, \ldots, n)$ are the elements of the matrix U, and $u_{ji} = u_{ij}$.

We shall consider the cases (a), (b) and (c) separately. In the case (a) we set $u_{i-1i} = -1$ $(i = 2, 3, \ldots, n)$, $u_{ij-1} + u_{i-1j} = 0$ $(i = 2, 3, \ldots, n-1;\; j = i+1, i+2, \ldots, n)$, $u_{1n} = u$. The successive principal diagonal minors Γ_k $(k = 1, 2, \ldots, n)$ of the resulting matrix B will have the form

$$\Gamma_k = 2pu + a_k p^2 + b_k p + c_k \quad (k = 1, 2, \ldots, n)$$

where a_k, b_k and c_k are pure numbers. If $a > 0$ and $b > 0$, then choosing $u > 0$ sufficiently large we obtain $\Gamma_k > 0$. If $a < 0$ and $b < 0$, then taking $u < 0$ sufficiently large in modulo we obtain once again $\Gamma_k > 0$. This, together with the Sylvester criterion, yields the positive definiteness of the matrix B.

In the case (b) we take $u_{i-1i} = -1$ $(i = 2, 3, \ldots, n/2, n/2 + 2, \ldots, n)$, $u_{ij-1} + u_{i-1j} = 0$ $(i = 2, 3, \ldots, n-1)$ $(j = i+1, \ldots, n)$, $u_{1n} = u$. Then $u_{n/2\, n/2+1} = -u$ and the successive principal diagonal minors of the matrix B will have the form

$$\Gamma_k = \begin{cases} 2pu + a_k p^2 + b_k p + c_k & (k = 1, 2, \ldots, n/2) \\ 4pu^2 + \sum_{i=1}^{2} (a_{ki} p^2 + b_{ki} p + c_{ki}) u^{2-i} & (k = n/2 + 1, \ldots, n) \end{cases}$$

where $a_k, b_k, c_k, a_{ki}, b_{ki}, c_{ki}$ are certain numbers. Choosing $u > 0$ sufficiently large, we obtain $\Gamma_k > 0$ and this ensures the positive definiteness of the matrix B.

In the case (c) we impose on the elements of the matrix U the restrictions used in the case (b) to obtain

$$u_{n/2\ n/2+1} = u$$

$$\Gamma_k = \begin{cases} 2pu + a_k p^2 + b_k p + c_k & (k = 1, 2, \ldots, n/2) \\ -4pu^2 + \sum_{i=1}^{2} (a_{ki} p^2 + b_{ki} p + c_{ki}) u^{2-i} & (k = n/2 + 1, \ldots, n) \end{cases}$$

where $a_k, b_k, c_k, a_{ki}, b_{ki}, c_{ki}$ are certain numbers. Taking $u < 0$ sufficiently large in modulo, we obtain $\Gamma_k > 0$. This completes the proof of the theorem.

Note. In the cases $n = 4k + 4$, $a < 0$, $b < 0$ and $n = 4k + 6$, $a > 0$, $b > 0$ $(k = 0, 1, 2, \ldots)$, Theorem 1 is not valid. Indeed, the equations

$$d^{4k+4}x / dt^{4k+4} - (2\pi / T)^{4k+4} x = 0$$

$$d^{4k+6}x / dt^{4k+6} + (2\pi / T)^{4k+6} x = 0$$

have infinitely many T-periodic solution.

Since under the conditions of Theorem 1 all requirements of Theorem II of [3] are satisfied for the system (2), the following corollary holds:

Corollary. Let $z^\circ(t)$ be a unique T-periodic solution of the system (2). Then manifolds M_1 and M_2 exist in the space E_n intersecting at the point $z^\circ(0)$ only, and are such that the following relations hold for the solutions $z(t)$ of the system (2):

$$\begin{aligned} &\|z(t) - z^\circ(t)\| \leqslant Ne^{-mt} \| z(0) - z^\circ(0) \|, \quad \text{if} \quad t \geqslant 0 \quad \text{and} \quad z(0) \in M_1 \\ &\|z(t) - z^\circ(t)\| \leqslant Ne^{mt} \| z(0) - z^\circ(0) \|, \quad \text{if} \quad t \leqslant 0 \quad \text{and} \quad z(0) \in M_2 \\ &\|z(t) - z^\circ(t)\| \geqslant Ke^{mt}, \quad \text{if} \quad t \geqslant t_0 \quad \text{and} \quad z(0) \overline{\in} M_1 \cup M_2 \end{aligned} \qquad (5)$$

where $N > 0$, $K > 0$, $m > 0$ and t_0 are constants.

Thus the unique T-periodic solution of the system (2) which is Liapunov unstable, is conditionally asymptotically stable to the right (left) of the maniford M_1 (M_2). Moreover, a nonlinear exponential dichotomy of solutions (see [3]) exists for the system (2).

Theorem 2. Let the following conditions hold for $t, x \in (-\infty, \infty)$.

1) the function $f(t, x)$ is continuous together with its derivative $\partial f / \partial x$, and T-periodic in t;

2) $f(t, 0) \equiv 0$.

Then the zero solution of the system (2) is unstable in each of the following cases:

a) $n = 2k + 3$ $(k = 0, 1, 2, \ldots)$, $\partial f(t, 0) / \partial x \neq 0$;

b) $n = 4k + 4$ $(k = 0, 1, 2, \ldots)$, $\partial f(t, 0) / \partial x > 0$;

c) $n = 4k + 6$ $(k = 0, 1, 2, \ldots)$, $\partial f(t, 0) / \partial x < 0$

Proof. Consider the following variational equation for the zero solution of the system (2):

$$dz / dt = (\partial F(t, 0) / \partial z) z \qquad (6)$$

All conditions of Theorem 1 hold for (6), therefore the last estimate of (5) which implies the positive definiteness of the characteristic Liapunov index of the solution $z(t)$ of (6), holds. As we know (see, e. g. [4]), the zero solution of the system (2) will in this case be unstable, and this proves the theorem.

Equation (1) was considered for $n \geqslant 3$. When $n = 2$, conditions (1)—(3) of Theorem 1 and the condition $a < 0$, $b < 0$ ensure the existence of a unique T-periodic solution of Eq. (1) and a nonlinear exponential dichotomy of the solutions of the system (2). A second order equation however, which is more general than (1), was studied in [1].

When $n = 1$, the conditions (1), (2) of Theorem 1 and the conditions

$$\partial f(t, x) / \partial x \geqslant a > 0 \tag{7}$$

$$\partial f(t, x) / \partial x \leqslant b < 0 \tag{8}$$

together ensure the existence of a unique T-periodic solution of Eq. (1). This solution is stable in the whole, and Eq. (1) represents a particular case of a monotonous differential equation studied in [2].

The author thanks A. I. Perov for the interest shown.

REFERENCES

1. Perov, A. I., Periodic, almost periodic and bounded solutions of the differential equation $dx/dt = f(t, x)$. Dokl. Akad. Nauk SSSR, Vol. 132, № 3, 1960.

2. Perov, A. I., Periodic Oscillations. Izd. Voronezh. Univ., 1973.

3. Perov, A. I. and Trubnikov, Iu. V., Monotonous differential equations. Tr. Inst. matem. i mekhan. Voronezh. Univ., 1973.

4. Demidovich, B. P., Lectures on the Mathematical Theory of Stability. Moscow, "Nauka", 1967.

Translated by L. K.